# Modelling the theoretical ELF spectra of lightning discharges with a continuing current

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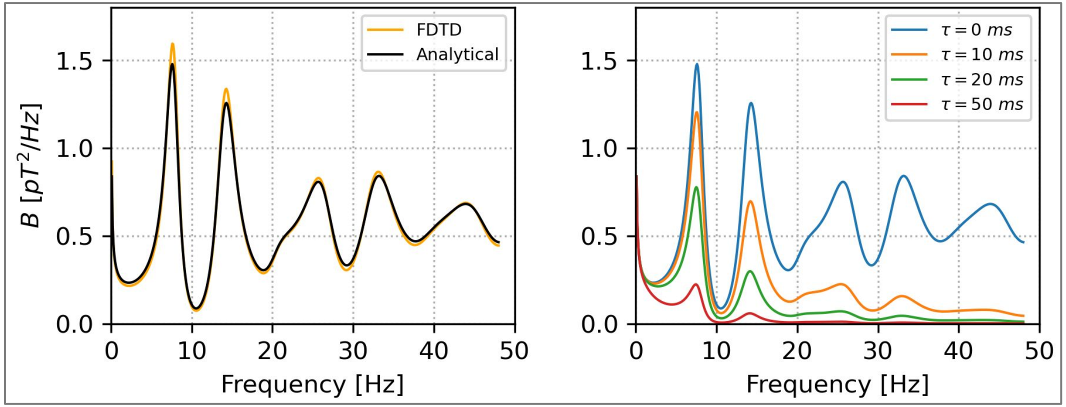


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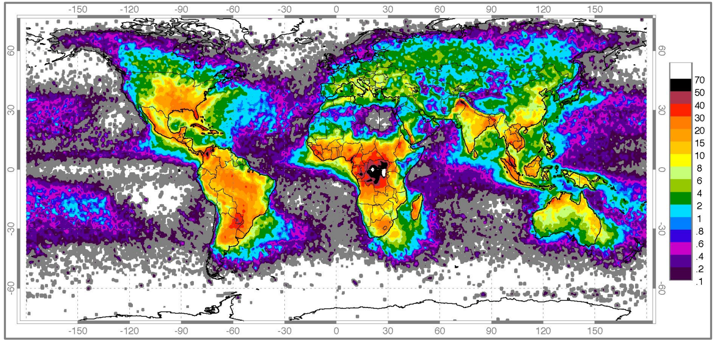
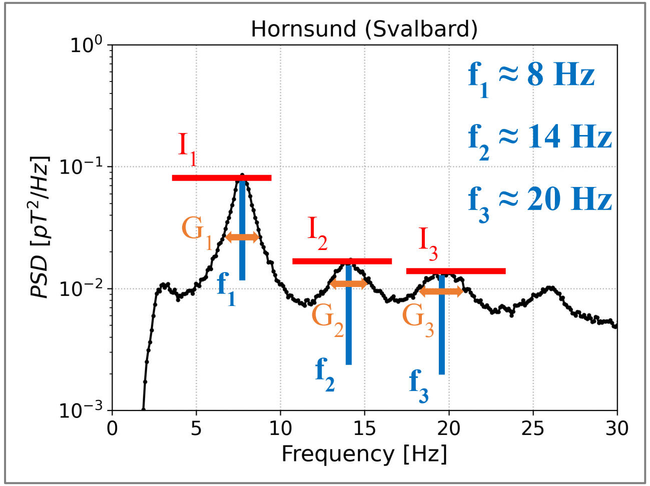
# Modelling Schumann resonances



**Forward modeling**

**Inversion**



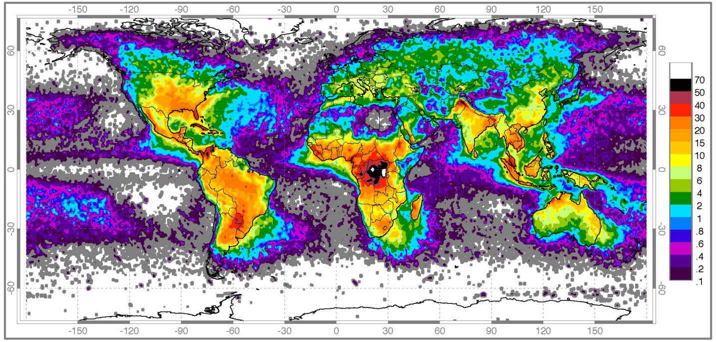

**Some benefits:**

- Independent view on global lightning activity
- Intensity of lightning activity in terms of an absolute physical quantity: vertical charge moment change
- Natural global integration

**Some challenges:**

- Lack of ground-truth lightning data
- Changing propagation conditions
- Model simplifications

# Modelling Schumann resonances



**Forward modeling**

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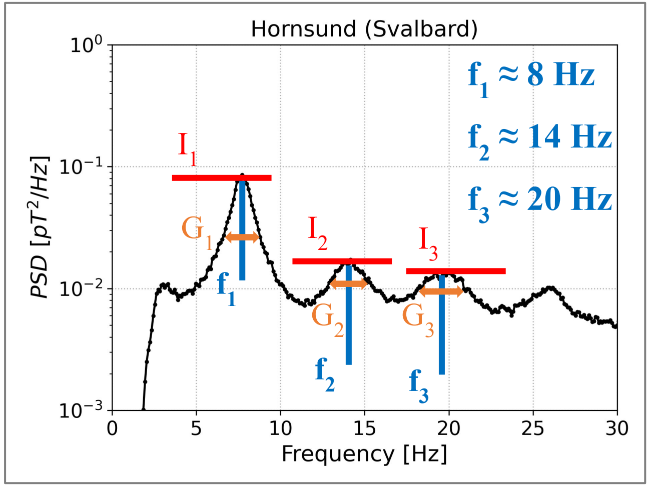



**Analytical model** following Kirillov et al. (1997) and Mushtak and Williams (2002):

$$E_r(\omega,\theta) = \frac{Ids(\omega)Z}{4\pi H_e^2}\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)+YZR^2}P_n(\cos\theta)\ ,$$

$$B_\phi(\omega,\theta) = \frac{\mu Ids(\omega)}{4\pi R H_e}\sum_{n=0}^{\infty}\frac{2n+1}{n(n+1)+YZR^2}P_n^1(\cos\theta)\ ,$$

**Ids(ω)**: Current moment spectrum of the source

Z(ω), Y(ω): Impedance and admittance parameters

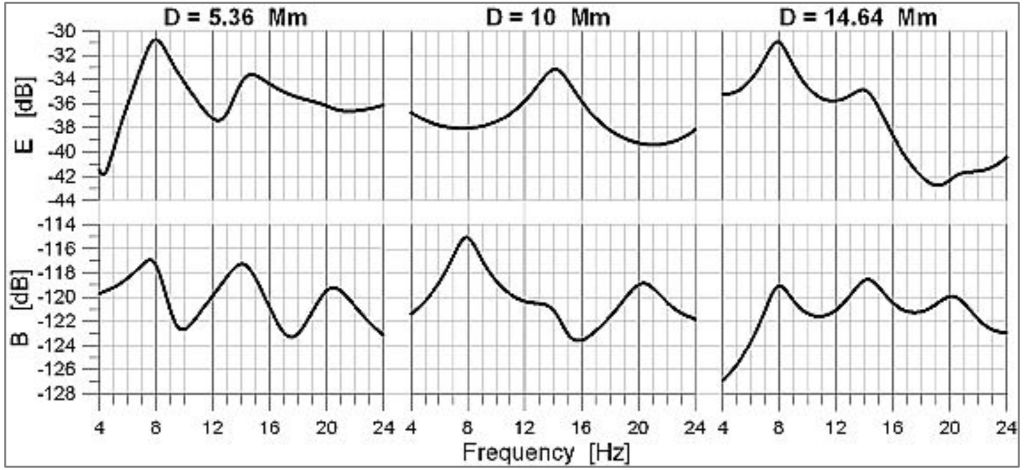
$H_e$(ω): Complex electric altitude of the waveguide

R: Earth's radius

$P_n$, $P_n^1$: Legendre and associated Legendre polynomials of order n

# Motivation (Kulak et al., 2006)

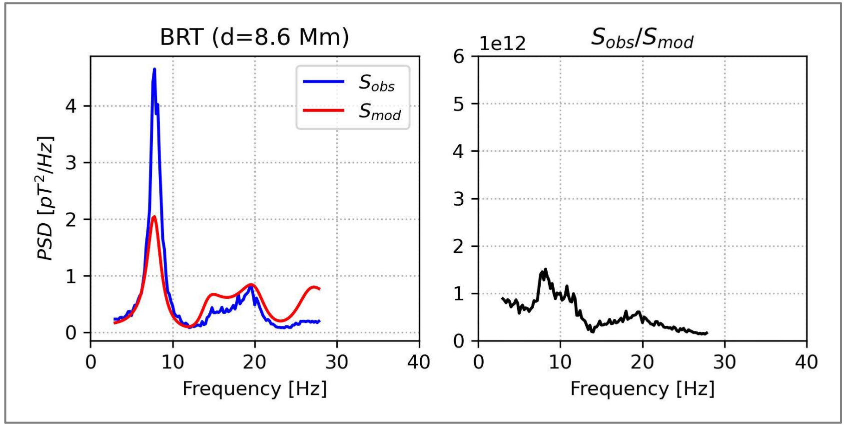
"The properties of damped resonators with a source are **more complex than it can be concluded from the classical solutions**. In this paper we present a model describing the ELF wave propagation inside the Earth-ionosphere cavity based on the assumption that the **ELF field in any point of the cavity is a superposition of a transmission and a resonance component** with phases dependent on the observer-source distance."

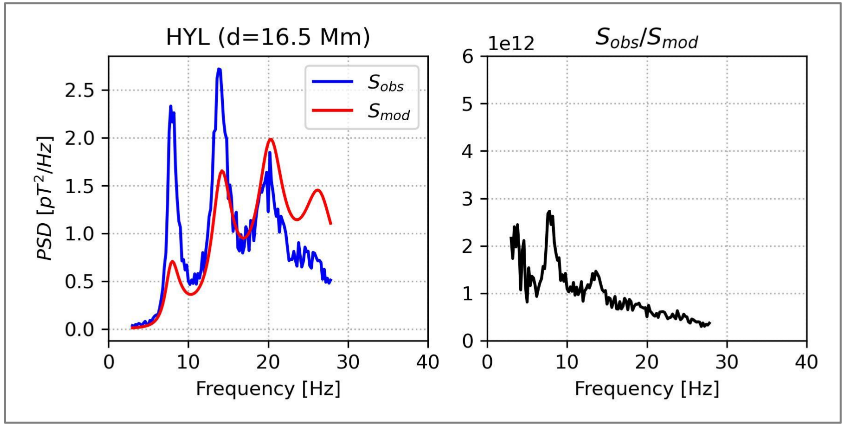



Amplitude spectra of electric E(ω) and magnetic B(ω) field components calculated for the observer-source distances 5.36, 10, and 14.64 Mm (from Kulak et al., 2006).

# Motivation (SR inversion)

It was already discussed by Madden & Thompson (1965) that the **SR source spectrum is not necessarily flat**. The presence of **"slow" discharges with a continuing current** and/or **correlations between repeated strokes** can enhance the low frequency end of the spectrum (Madden & Thompson, 1965). Shvets (2001) found that the theoretical SR spectra best fit the measurements when an **effective time constant** of 3.5 ms was used.




Reconstruction of ELF spectra measured at Bharati (Antarctica) and Hylaty (Poland) stations during the eruption of the Hunga-Tonga volcano in 2022.

# Main research questions

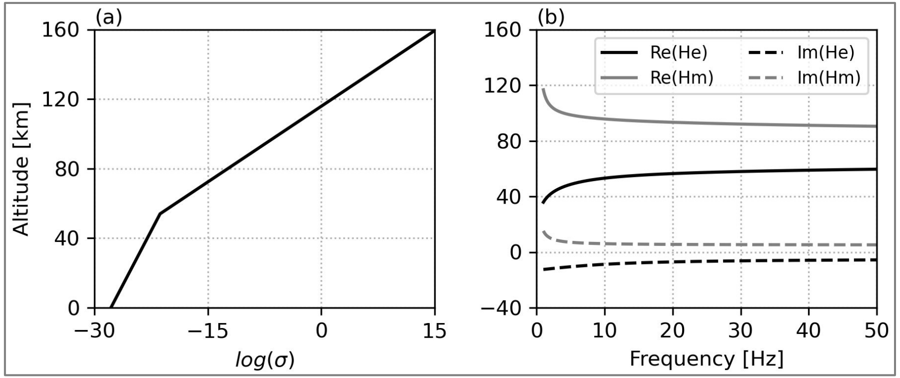
- Does the analytical model accurately describe the superposition of propagating and traveling waves in the Earth-ionosphere cavity resonator?
- Can we use a modified version of the analytical model to "redden" the theoretical ELF spectra to get them closer to the measurements?

**Full numerical (FDTD) model** from Marchenko et al. (2022)

$$\varepsilon_0 \frac{\partial E_r}{\partial t} + \sigma E_r + J_r = \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}\left(H_\phi \sin\theta\right)\right], \quad (1)$$

$$\varepsilon_0 \frac{\partial E_\theta}{\partial t} + \sigma E_\theta = -\frac{1}{r}\frac{\partial}{\partial r}\left(r H_\phi\right), \quad (2)$$

$$\mu_0 \frac{\partial H_\phi}{\partial t} = -\frac{1}{r}\left[\frac{\partial}{\partial r}\left(r E_\theta\right) - \frac{\partial E_r}{\partial \theta}\right], \quad (3)$$

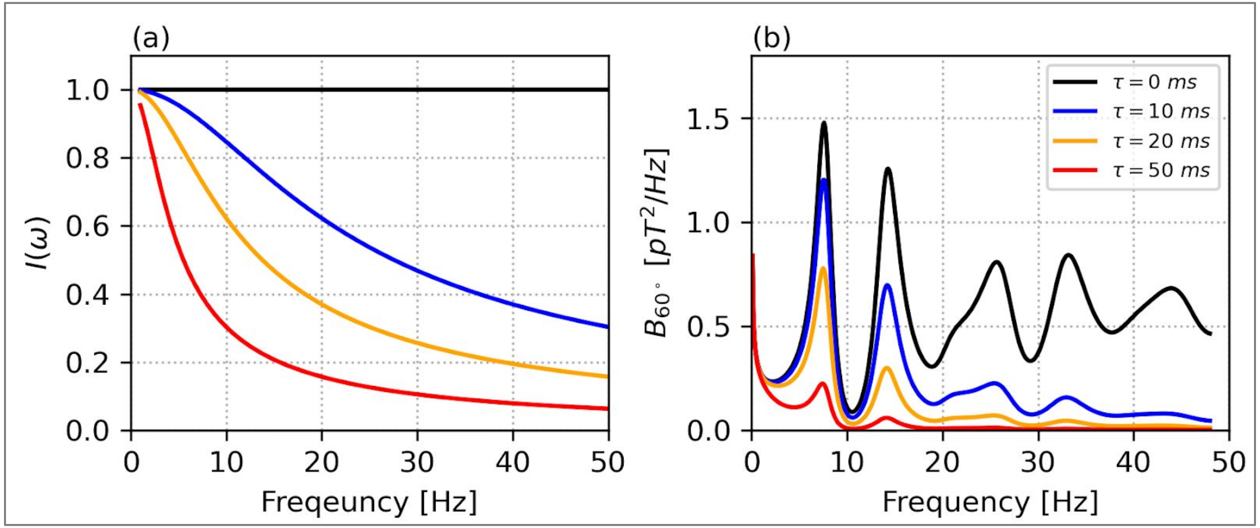



**(a)** The "knee" conductivity profile used in the FDTD model and **(b)** the complex, frequency dependent He and Hm altitudes determined from this profile and used in the analytical calculations (Mushtak & Williams, 2002).

# Source model

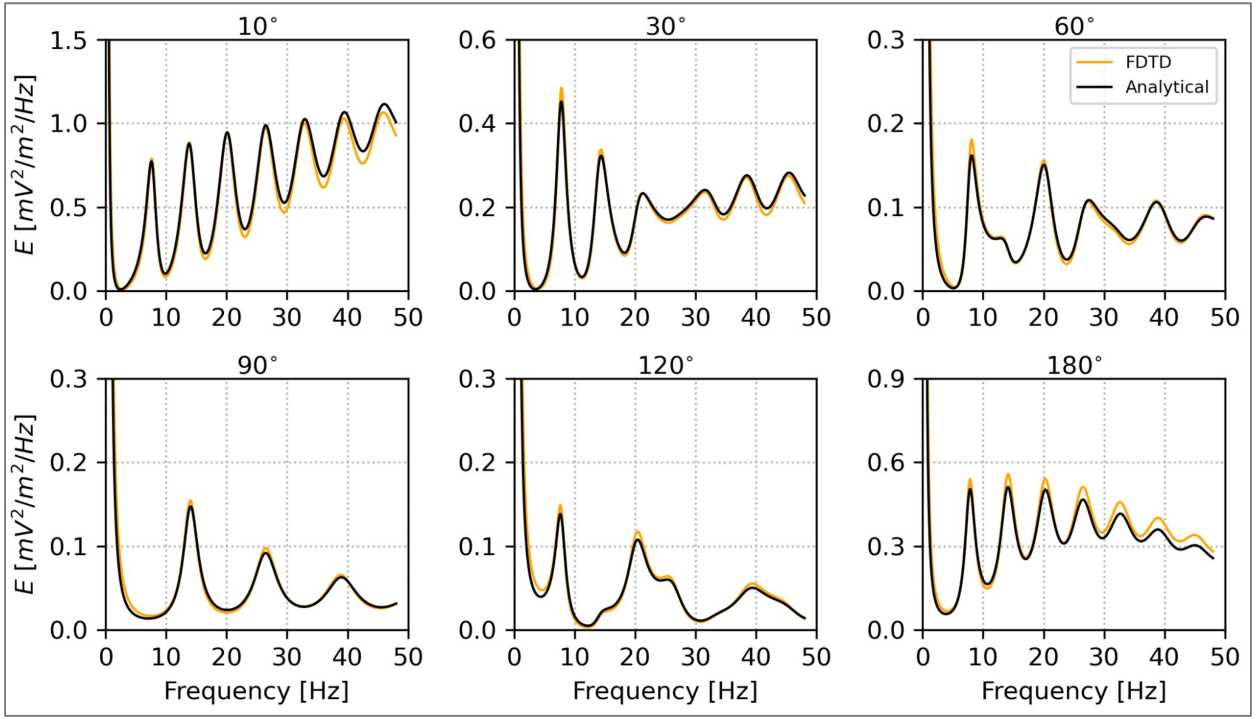
Time domain: $I(t) = \begin{cases} 0 & t < 0, \\ \frac{1}{\tau} e^{-t/\tau} & t \geq 0. \end{cases}$

Frequency domain: $I(\omega) = \dfrac{1}{1 + i\omega\tau}$ .



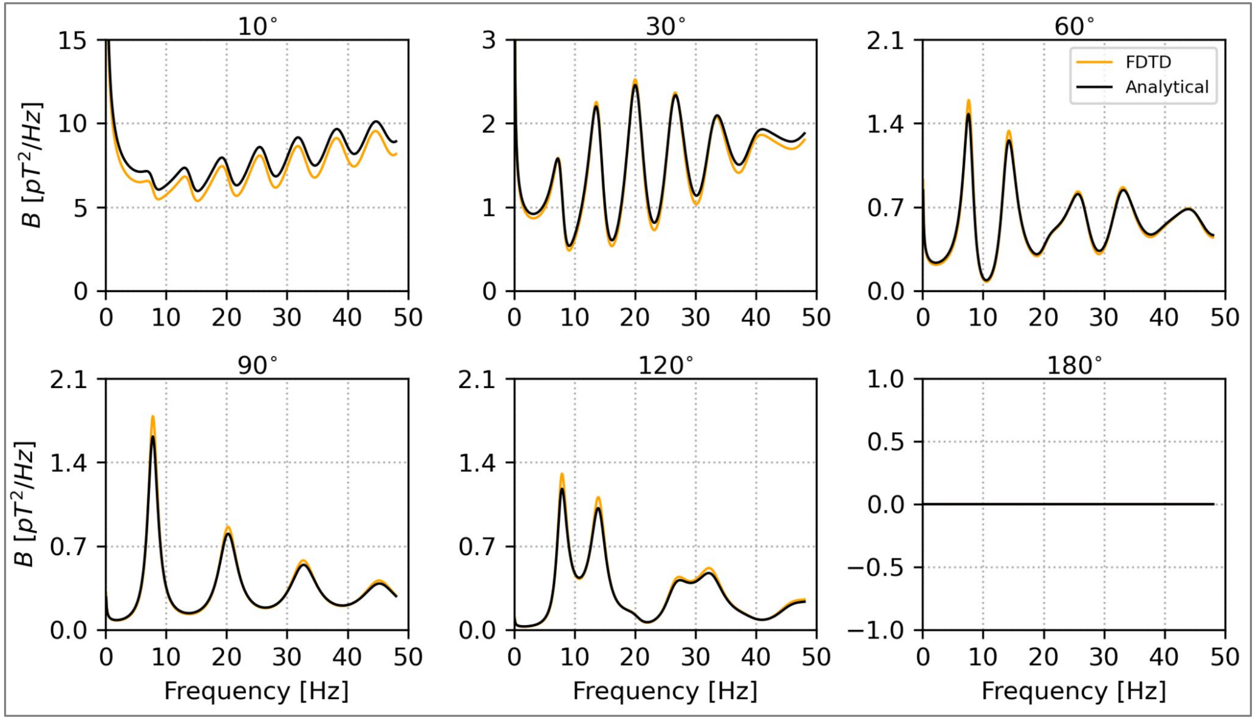

**(a)** Current spectra of lightning sources with different decay times (τ). **(b)** Theoretical magnetic spectra of 60° for the same τ values calculated with the analytical model. Note the more and more reddish spectrum with increasing τ.

# Results




Theoretical **electric** spectra for different source-observer distances (10º, 30º, 60º, 90º, 120º, 180º) calculated with the analytical (black) and the full numerical (FDTD) model (gray) with an **impulse-like excitation source** (τ = 0 ms).

# Results




Theoretical **magnetic** spectra for different source-observer distances (10º, 30º, 60º, 90º, 120º, 180º) calculated with the analytical (black) and the full numerical (FDTD) model (gray) with an **impulse-like excitation source** ($\tau$ = 0 ms).

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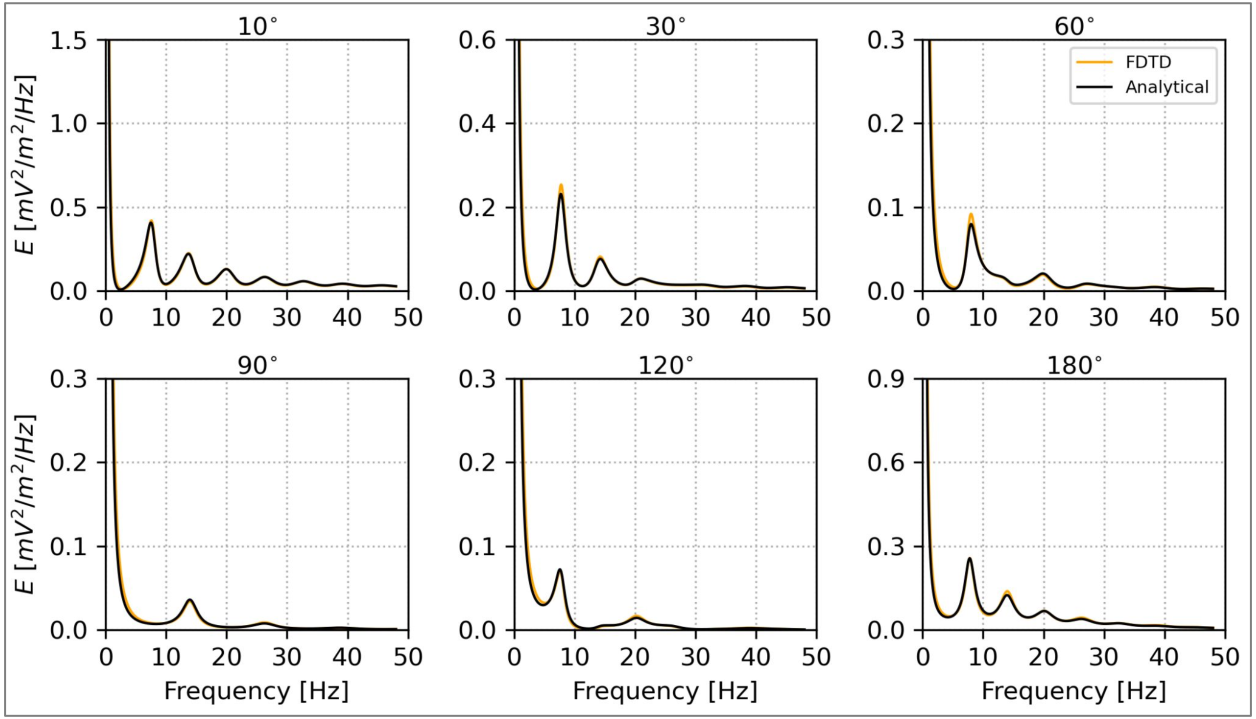


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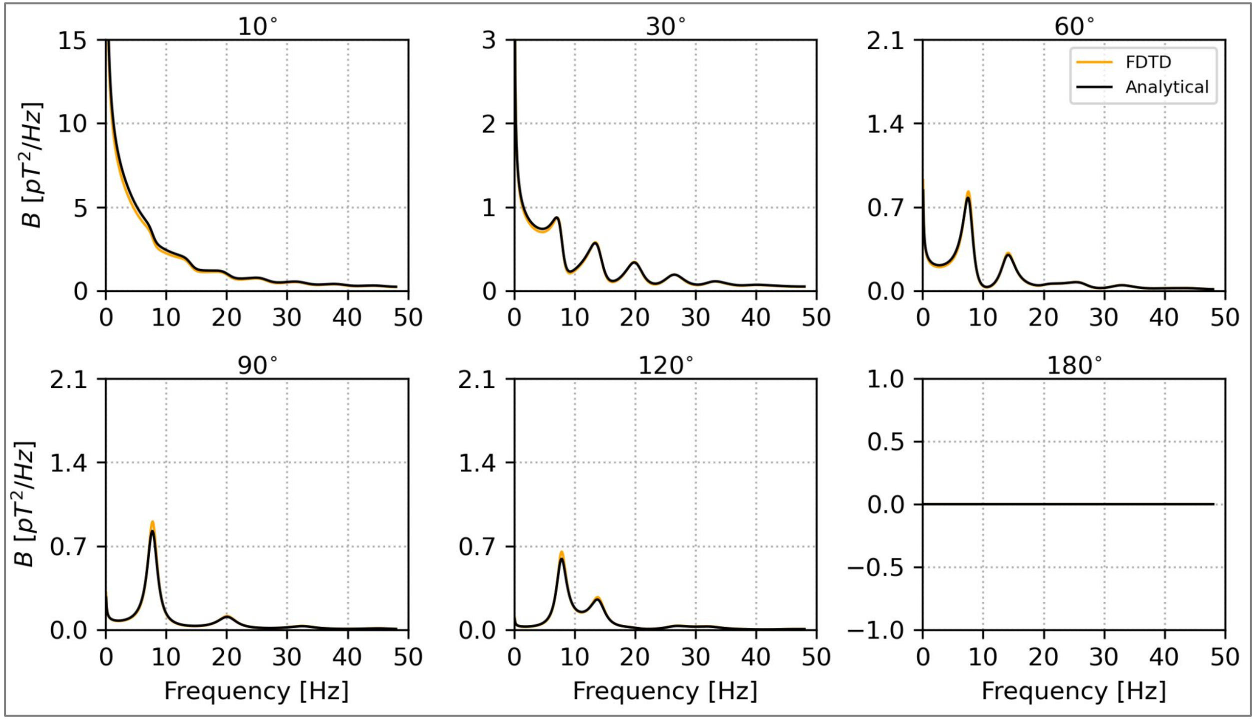



Theoretical **magnetic** spectra for different source-observer distances (10º, 30º, 60º, 90º, 120º, 180º) calculated with the analytical (black) and the full numerical (FDTD) model (gray) with an **exponentially decaying excitation source** (τ = 20 ms).

| Distance [º] | $C_{E, delta}$ | $C_{E, 10ms}$ | $C_{E, 20ms}$ | $C_{E, 50ms}$ |
|---|---|---|---|---|
| 10 | 0.998 | 0.997 | 0.996 | 0.996 |
| 30 | 0.997 | 0.998 | 0.998 | 0.999 |
| 60 | 0.995 | 0.998 | 0.998 | 0.992 |
| 90 | 0.996 | 0.995 | 0.992 | 0.965 |
| 120 | 0.998 | 0.996 | 0.996 | 0.993 |
| 180 | 0.997 | 0.994 | 0.996 | 0.992 |
| **Distance [º]** | $C_{B, delta}$ | $C_{B, 10ms}$ | $C_{B, 20ms}$ | $C_{B, 50ms}$ |
| 10 | 0.998 | 1.000 | 1.000 | 1.000 |
| 30 | 0.999 | 0.998 | 0.999 | 0.999 |
| 60 | 0.998 | 0.998 | 0.998 | 0.998 |
| 90 | 0.999 | 0.999 | 0.999 | 0.999 |
| 120 | 0.999 | 0.999 | 0.999 | 0.999 |

The **correlation coefficients** ($C_E$ and $C_B$) between theoretical spectra (5-30 Hz) calculated with the analytical and the full numerical models for different source-observer distances and excitation sources.

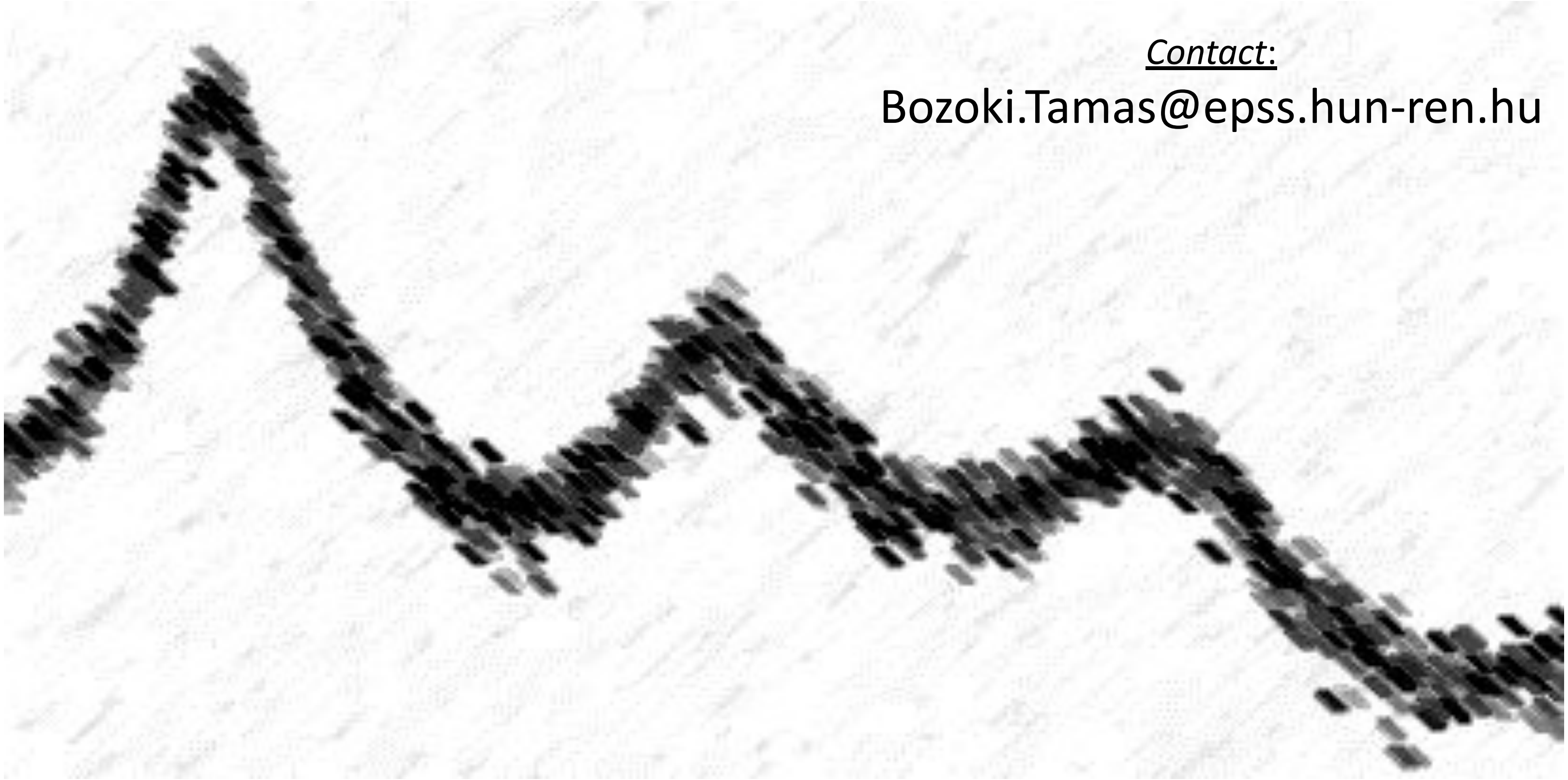
| Distance [º] | $\Delta S_{E,delta}$ [%] | $\Delta S_{E,10ms}$ [%] | $\Delta S_{E,20ms}$ [%] | $\Delta S_{E,50ms}$ [%] |
|---|---|---|---|---|
| 10 | 8.2 | 6.6 | 5.8 | 4.2 |
| 30 | 5.3 | 4.8 | 5.2 | 8.2 |
| 60 | 5.6 | 7.6 | 11.9 | **24.1** |
| 90 | 8.9 | 7.9 | 8.7 | 14.1 |
| 120 | 10.8 | 12.2 | 15.4 | **25.1** |
| 180 | 4.9 | 4.5 | 4.5 | 8.7 |
| **Distance [º]** | **$\Delta S_{B,delta}$ [%]** | **$\Delta S_{B,10ms}$ [%]** | **$\Delta S_{B,20ms}$ [%]** | **$\Delta S_{B,50ms}$ [%]** |
| 10 | 8.3 | 8.1 | 7.6 | 5.9 |
| 30 | 5.0 | 5.1 | 5.0 | 4.7 |
| 60 | 4.5 | 4.7 | 4.8 | 5.1 |
| 90 | 3.5 | 3.6 | 3.7 | 3.8 |
| 120 | 3.7 | 3.6 | 3.5 | 3.5 |

The **mean relative differences** ($\Delta S_E$ and $\Delta S_B$) between theoretical spectra (5-30 Hz) in % calculated with the analytical and the full numerical models for different source-observer distances and excitation sources.

# Summary

- The global EM resonance field produced by impulse-like and exponentially decaying lightning sources can be described with **good accuracy** by a modified version of the widely used analytical model describing Schumann resonances.
- This result indicates that the stationary solution given by the **analytical model is able to describe the superposition of standing and traveling waves** in the strongly damped Earth-ionosphere cavity resonator.
- For future studies dealing with "background" SRs, **we propose the use of the exponentially decaying excitation source in the analytical model**, which may lead to a better agreement with the measurements than the impulse-like excitation source.

# Thank you for your attention!



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